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The Waldspurger Transform of Permutations and Alternating Sign Matrices

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Abstract. In 2005 J. L. Waldspurger proved the following theorem: Given a finite real reflection group G, the closed positive root cone is tiled by the images of the open weight cone under the action of the linear transformations $1 - g$. Shortly after this E. Meinrenken extended the result to affine Weyl groups and then P. V. Bibikov and V. S. Zhgoon gave a uniform proof for a discrete reflection group acting on a simply-connected space of constant curvature.

In this paper we show that the Waldspurger and Meinrenken theorems of type A give an interesting new perspective on the combinatorics of the symmetric group. In particular, for each permutation matrix $g \in \mathfrak{S}_n$ we define a non-negative integer matrix $\mathbf{WT}(g)$, called the *Waldspurger transform* of g. The definition of the matrix $\mathbf{WT}(g)$ is purely combinatorial but it turns out that its columns are the images of the fundamental weights under $1 - g$, expressed in simple root coordinates. The possible columns of $\mathbf{WT}(g)$ (which we call *UM vectors*) biject to many interesting structures including: unimodal Motzkin paths, abelian ideals in the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, Young diagrams with maximum hook length n, and integer points inside a certain polytope.

We show that the sum of the entries of $\mathbf{WT}(g)$ is half the entropy of the corresponding permutation g, which is known to equal the rank of g in the MacNeille completion of the Bruhat order. Inspired by this, we extend the Waldspurger transform $\mathbf{WT}(M)$ to alternating sign matrices M and give an intrinsic characterization of the image. This provides a geometric realization of MacNeille completion of the Bruhat order (a.k.a. the lattice of alternating sign matrices).

Keywords: Alternating Sign Matrices, Waldspurger Transform, Affine Symmetric Group, Abelian Ideals, SIF Permutations, Bruhat Order, MacNeille Completion

1 Waldspurger and Meinrenken Theorems

Our work is motivated by making the following theorems explicit for type A, where the finite Weyl group is the symmetric group, $\mathfrak{S}_n$ and the affine Weyl group is $\tilde{\mathfrak{S}}_n$.

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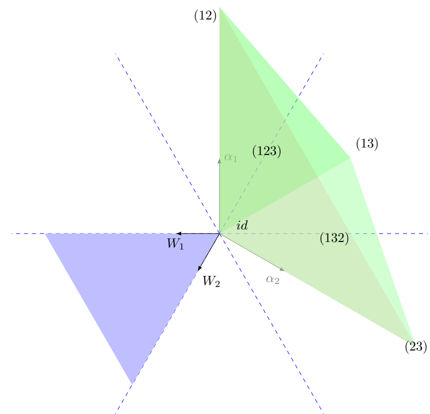


Figure 1: The Waldspurger Decomposition for $A_2 = \mathfrak{S}_3$

Theorem 1 (J. L. Waldspurger, 2005 [7])**.** *Let W be a Weyl group presented as a reflection group on a Euclidean vector space V. Let $C_w \subset V$ be the open cone over the fundamental weights and $C_R \subset V$ the closed cone spanned by the positive roots. Let the cone associated with group element g be $C_g := (I - g)C_w$ (where I is the identity element in G). One has the decomposition*

$$C_R = \bigsqcup_{g \in W} C_g.$$

Theorem 2 (E. Meinrenken, 2006 [5, 2])**.** *Let the affine Weyl group for a crystallographic Coxeter system be denoted W^a and recall that $W^a = \Lambda \rtimes W$ where the co-root lattice $\Lambda \subset V$ acts by translations. Let $A \subset C$ denote the Weyl alcove, with $0 \in A$. Then the images $V_g = (id - g)A$, $g \in W^a$ are all disjoint, and their union is all of V. That is,*

$$V = \bigsqcup_{g \in W^a} V_g.$$

We will define the **Meinrenken tile** to be $\bigsqcup_{g \in W} V_g$, restricting to a copy of the finite Weyl group inside of the affine Weyl group. The semi-direct product with the co-root lattice simply translates the Meinrenken tile and so this restriction is convenient from a combinatorial perspective. Although it is built out of simplices, the Meinrenken tile is not a simplicial complex, or even a CW complex, and it need not even be convex.

In type A, both C_g and V_g are easily computed in terms of what we call the *Waldspurger Matrix of the permutation g*, $\mathbf{WT}(g)$. We will define these matrices in the next section and show how to compute them for a given permutation.

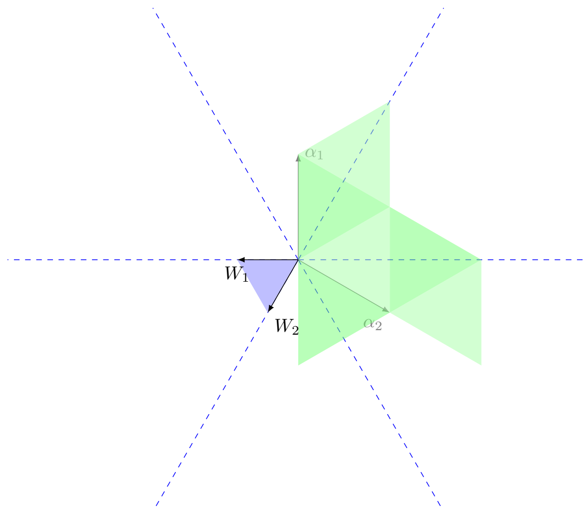


Figure 2: The Meinrenken tiling for $A_2 = \mathfrak{S}_3$.

2 The Waldspurger Transform for Permutations

Definition 1. *Let ϕ denote the reflection representation of the symmetric group*

$$\phi : \mathfrak{S}_n \longrightarrow GL_{n-1}(\mathbb{R}).$$

The Waldspurger matrix, $\mathbf{WT}(g)$, of a permutation g is the matrix of $\phi(1) - \phi(g)$ applied to the matrix with columns the fundamental weights, all expressed in root coordinates.

Our first theorem gives a concrete combinatorial way of finding the Waldspurger matrix associated with a given permutation. It is helpful to review the definition of the Cartan Matrix of a root system and note what it looks like for type A. The *Cartan matrix* of a root system is the matrix whose elements are the scalar products

$$a_{ij} = 2\frac{(r_i, r_j)}{(r_i, r_i)}$$

(sometimes called the Cartan integers) where the r_i's are the simple roots. Recall that the root system A_{n-1} has simple roots the vectors $a_i : e_i - e_{i+1}$ for $i = 1, \ldots, n-1$. One can verify that the Cartan matrix for this root system has two's on the main diagonal, negative one's on the superdiagonal and subdiagonal, and zeros elsewhere. Its columns express the simple roots in the basis of fundamental weights.

Theorem 3. *Let P be the $(n-1) \times (n-1)$ matrix for the permutation $\pi \in \mathfrak{S}_n$ expressed in root coordinates. Let C be the $(n-1) \times (n-1)$ Cartan matrix and let D be the $(n-1) \times (n-1)$ matrix*

$$D_{i,j} = \begin{cases} \displaystyle\sum_{\substack{a \le i \\ b > j}} \pi_{a,b} & i \le j \\ \displaystyle\sum_{\substack{a > i \\ b \le j}} \pi_{a,b} & i \ge j. \end{cases}$$

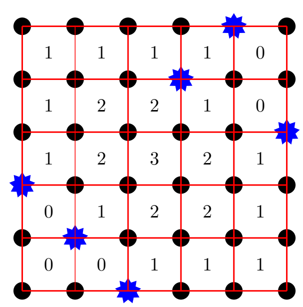
Then we have that $(\mathbf{I} - \mathbf{P}) = \mathbf{DC}$*.*

Because the inverse of the Cartan matrix expresses the fundamental weights in simple root coordinates, we may multiply both sides of the equation above on the right by $\mathbf{C}^{-1}$ and observe

$$\mathbf{D} = \mathbf{WT}(\pi).$$

Computing the Waldspurger Transform Combinatorially:

Let $\pi \in \mathfrak{S}_n$ be expressed as an $n \times n$ permutation matrix. For aesthetics our examples put the entries of π on a grid, leave off the zeros, and use stars instead of ones. Construct the $n-1 \times n-1$ Waldspurger matrix $\mathbf{WT}(\pi)$ in the spaces between the entries of the permutation matrix as follows: If an entry is on or above the main diagonal, count the number of stars above and to the right, and put it in the space. If the entry is on or below the main diagonal, count the number of stars below and to the left and put it in the space. Note that entries on the diagonal are still well defined. As an example, here is the Waldspurger matrix for the permutation $456213 \in \mathfrak{S}_6$.



Now suppose M is a Waldspurger matrix for the permutation π, with columns $c_1, c_2, \ldots, c_{n-1}$. To return to the language of the Waldspurger and Meinrenken theorems we have:

$$C_M := C_\pi = \left\{ \sum_{i=1}^{n-1} a_i c_i \middle| \quad a_i \in \mathbb{R}_{\ge 0} \right\} \tag{2.1}$$

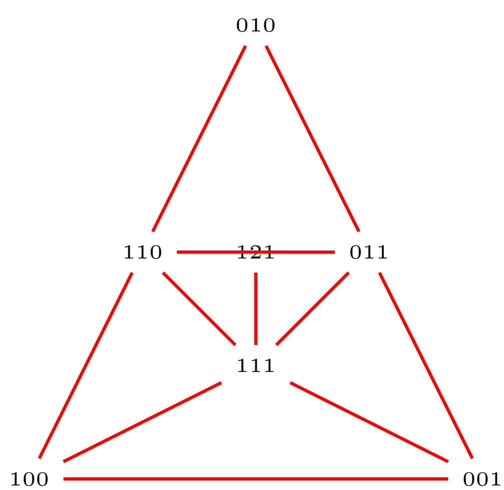
$$V_M := V_\pi = \left\{ \sum_{i=1}^{n-1} a_i c_i \middle| \quad a_i \in \mathbb{R}_{\ge 0} \text{ and } \sum a_i \le 1 \right\}. \tag{2.2}$$

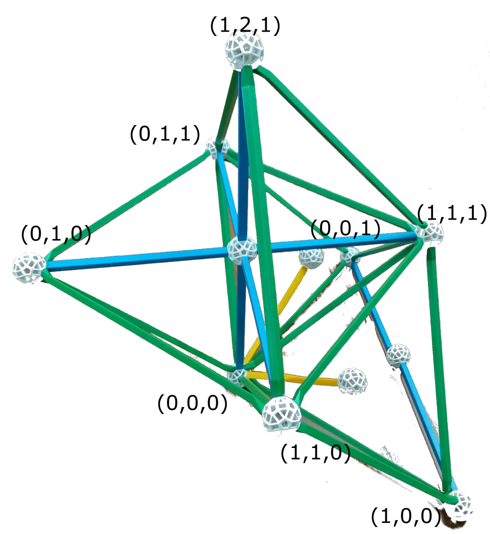
It is at times convenient to study the boundary of the Meinrenken tile, so we will also define

$$\Delta_M := \Delta_\pi := \left\{ \sum_{i=1}^{n-1} a_i c_i \middle| \quad a_i \in \mathbb{R}_{\geq 0} \text{ and } \sum a_i = 1 \right\}. \tag{2.3}$$

3 Geometric Observations... what is this thing?

Our first example, in Figures 1 and 2, was in many ways too nice. One may be tempted to study the Meinrenken tile or a slice of the root cone as a simplicial complex, or at the very least a regular CW complex. Going up even one dimension presents several unforeseen complications. For starters, our Meinrenken tile is no longer convex! Below a slice of the root cone in the Waldspurger decomposition for $W = \mathfrak{S}_4$, labeled with important points in root coordinates, appears on the left, and the corresponding Meinrenken tile constructed out of zometools on the right. (The two yellow edges and one blue edge coming out from the origin are the fundamental weights.)





One can now see where the slice of the root cone and the Meinrenken tile fail to be simplicial or regular CW complex. In the Waldspurger picture, the top triangle intersects the two below it along "half edges". One may desire to consider it instead as a degenerate square to fix this impediment, but from the Meinrenken tile, it seems this new vertex should rightly be the fundamental weight with root coordinates $(\frac{1}{2}, 1, \frac{1}{2})$ and not the vertex $(1, 2, 1)$. If we wish to proceed in this manner, we must then include $(\frac{1}{2}, 1, \frac{1}{2})$ as a vertex for the two triangles 110,121,111 and 111,121,011 and consider them as degenerate tetrahedra. This sort of topological completion via intersecting facets has proven to be a rabbit hole with less fruit than one might hope for. Perhaps some insight can be lent

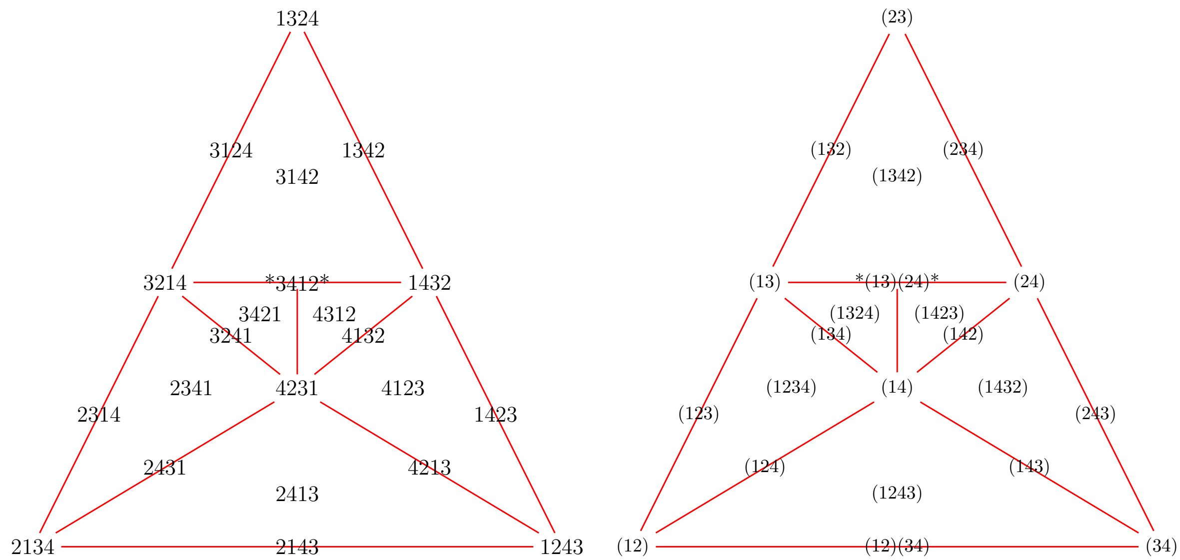


Figure 3: A slice of the Root cone $A_3 = \mathfrak{S}_4$

from the theory of simplicial sets. Instead let us turn our attention back to the symmetric group, and consider Figure 3.

Observe that the dimension of a simplex in the cone slice relates to the number of cycles (counting fixed points as one cycles) of the corresponding permutation. The four cycles are the triangles, the three cycles and disjoint two cycles are the edges, and the transpositions are vertices. This has been known for some time and can be seen as a corollary to the Chevalley-Shephard-Todd theorem. The astute observer will notice that there are two permutations missing in the picture. The identity corresponds to the cone point which we cut off, and the vertical edge in the center we left unlabeled, as we feel that (along with the starred edge 3412) it deserves some discussion. It corresponds to the permutation 4321 and its 3×3 Waldspurger matrix has all entries equal to one except for a two in the middle. If we consider the columns of each Waldspurger matrix as being ordered from left to right, the cones in the Waldspurger decomposition are endowed with an orientation. The orientation appears to be consistent, but what does it say in the case of this permutation? It appears to go first up from $(1,1,1)$ to $(1,2,1)$ and then back down. The starred edge, 3412 is also strange. Its Waldspurger matrix has first column $(1,1,0)$ second column $(1,2,1)$ and third column $(0,1,1)$ so it is perhaps better seen as a degenerate triangle than as an edge. Looking at the Meinrenken tile, we see that $\Delta_{3,4,1,2}$ is actually a triangle. The strangeness in the Meinrenken picture comes from the fact that $V_{3,4,1,2}$ is a square and not a tetrahedron.

Despite all of these collapses in dimension, there is still a fair amount of symmetry in the Meinrenken tile.

Theorem 4. *Let θ denote the longest positive root, i.e. the vector of all ones in root coordinates.*

Then reflection through the affine hyperplane orthogonal to θ with height one is an involution on the set of Δ_π's.

At the level of permutations, this involution is just applying the transposition $(1, n)$ on the left. In contrast, applying the transposition $(1, n)$ on the right is the gluing map for using multiple Meinrenken tiles to tile space. The left right symmetry is conjugation by the longest element in the Coxeter group.

4 Motzkin Paths, Abelian Ideals, Tableaux, and More

Definition 2. *A* UM vector *is any vector that appears as a column in* $\mathbf{WT}(\pi)$ *for some permutation* π*.*

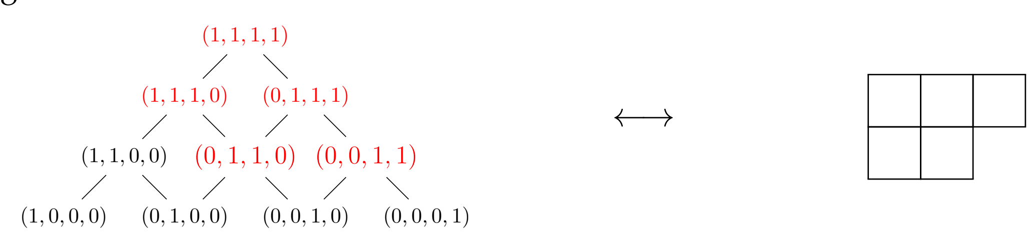
Theorem 5. *A UM vector must start with a zero or a one, weakly increase by one until its entry on the diagonal, and then weakly decreases by one until its final entry, a zero or one. Any row vector of a Waldspurger matrix must also be a UM vector with its maximum also on the diagonal. There are* 2^n *UM vectors of length* n*.*

Definition 3. *A* Motzkin path *is a lattice path in the integer plane* $\mathbb{Z} \times \mathbb{Z}$ *consisting of steps* $(1, 1), (1, -1), (1, 0)$ *which starts and ends on the x-axis, but never passes below it. A Motzkin path is* unimodal *if all occurrences of the step* $(1, 1)$ *are before the occurrences of* $(1, -1)$*. For brevity, we refer to unimodal Motzkin paths as* UMP*'s.*

Theorem 6. *There is a bijective correspondence between UM vectors of length* $n - 1$ *and UMPs with* n *steps.*

Theorem 7. *UM vectors are in bijection with tableaux with hook length bounded above by* n *and with Abelian ideals in the nilradical of the Lie Algebra* $\mathfrak{sl}_n$*.*

One can take any UM vector and write it as a sum of positive roots by recursively subtracting the highest root whose nonzero entries correspond to positive nondecreasing entries in the UM vector. For example, the vector $(0, 1, 2, 1) = (0, 1, 1, 0) + (0, 0, 1, 1)$ This set of positive roots will always generate an abelian ideal in the nilradical of the Lie Algebra $\mathfrak{sl}_n$ and will correspond to a tableau with bounded hook length, as seen in the diagram below.



Theorem 8. *UM vectors are exactly the coroots c (in root coordinates) such that* $-1 < (c, r) < 2$ *for every positive root r. These are the coroots inside the polytope defined by affine hyperplanes at heights negative one and two orthogonal to every positive root.*

This follows from a result which Panyushev attributes to Peterson and Kostant [6] which is expressed in the language of Abelian ideals.

5 Entropy, Alternating Sign Matrices, and the Waldspurger Transform in General

Definition 4. *The Entropy (alternatively called variance in the literature) of a permutation* π*, is*

$$E(\pi) := \sum_{i=1}^{n} (\pi(i) - i)^2.$$

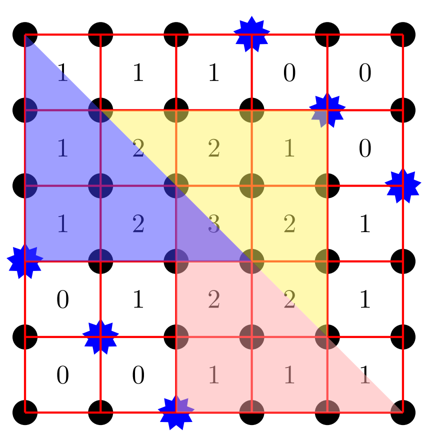
Definition 5. *The Waldspurger Height of a permutation* π*, is*

$$h(\pi) := \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{WT}(\pi)_{i,j}.$$

Theorem 9. *For* $\pi \in \mathfrak{S}_n$*,*

$$h(\pi) = \frac{1}{2} E(\pi).$$

Proof. Consider what each "star" in the transformation diagram contributes to the Waldspurger matrix. We can see it as contributing ones to every entry enclosed in the right triangle between itself and the main diagonal, and contributing one half for every entry on the main diagonal whose box is cut by the triangle.



$\square$

Definition 6. Alternating Sign Matrices *or ASMs, are square matrices with entries* 0, 1, *or* -1 *whose rows and columns sum to* 1 *and alternate in sign.*

Theorem 10 (A. Lascoux and M. Schützenberger, 1996)**.** *One half the entropy of a permutation is its rank in the Dedekind-MacNeille completion of the Bruhat order. The elements in this lattice can be viewed as alternating sign matrices with partial order component-wise comparison of entries in their associated monotone triangles.*

The Dedekind-MacNeille completion of a poset P is defined to be the smallest lattice containing P as a subposet [3]. Its construction is similar to the Dedekind cuts used to construct the real numbers from the rationals. For more on alternating sign matrices, monotone triangles, and their history we refer to [4]. This connection to alternating sign matrices motivates us to define the **Waldspurger Transform** of a matrix, not just a permutation.

Definition 7. *An $n \times n$ matrix M is sum-symmetric if its ith row sum equals its ith column sum for all $1 \leq i \leq n$. We write $M \in SS_n$.*

Definition 8. *From an $n \times n$ sum-symmetric matrix M, define the $n-1 \times n-1$ matrix, $\mathbf{WT}(M)$ where*

$$\mathbf{WT}(M)_{i,j} = \begin{cases} \sum\limits_{\substack{a \leq i \\ b > j}} M_{a,b} & i \leq j \\ \sum\limits_{\substack{a > i \\ b \leq j}} M_{a,b} & i \geq j \end{cases}.$$

Proposition 1. $\mathbf{WT}(M)$ *is well-defined if and only if $M \in SS_n$. If M were not sum-symmetric, the diagonal would be "over-determined."*

Proposition 2. *The* **WT** *map is linear and surjective with kernel the diagonal matrices.*

$$\mathbf{WT} : SS_n \twoheadrightarrow \mathrm{Mat}_{n-1}$$

Theorem 11. *The restriction of the Waldspurger transform to the alternating sign matrices has as its image all $M \in \mathrm{Mat}_{n-1}$ such that columns and rows of M are UM vectors with maximums on the diagonal. Component-wise comparison of these matrices is exactly the same order as is defined on the ASM lattice via monotone triangles.*

We may now extend many of the previous definitions to consider not just permutations, but alternating sign matrices. In particular, we consider V_M and Δ_M, simplices corresponding to ASMs, and h_M, the Waldspurger height of an ASM.

Theorem 12. *The height statistic not only gives the rank of an ASM M in the lattice, it also is a literal height of the baricenter of Δ_M inside of the Meinrenken tile in the direction of the sum of the positive roots. This gives a geometric realization of the ASM lattice inside the Meinrenken tile, with the caveat that baricenters of Δ_{M_1} and Δ_{M_2} may coincide if both M_1 and M_2 are not permutations.*

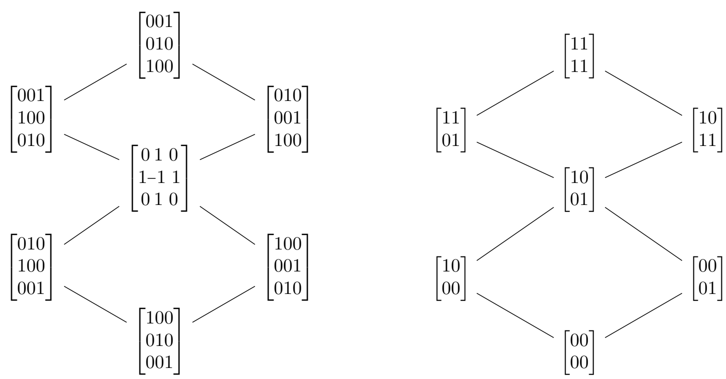


Figure 4: The Dedekind MacNeille completion of the Bruhat order for $A_2 = \mathfrak{S}_3$ and the corresponding Waldspurger matrices.

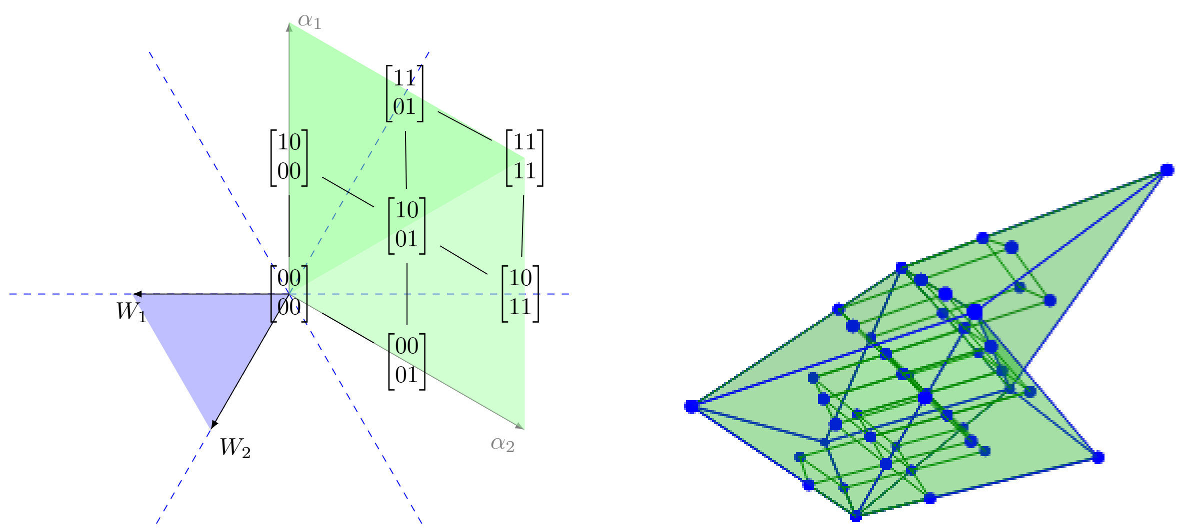


Figure 5: Place $\mathbf{WT}(M)$ at the baricenter of Δ_M for each $M \in ASM$ to get a geometric realization of the Hasse diagram inside the Meinrenken tile. (Four of the baricenters in the A_3 picture have multiplicity two.)

5.1 SIF Permutations and Dimensionality

A recent connection has been made between the connected positroids of Postnikov introduced to study the totally nonnegative Grassmannian and SIF permutations [1]. Here we show that SIF permutations are exactly the ones with distinct non zero columns in their Waldspurger matrices. It may be a worthwhile endeavor to study relationships among connected positroids via the geometry of the Meinrenken tile. To that end, we state the following relevant theorems:

Definition 9. *A permutation on* $[n] = \{1, 2, \ldots, n\}$ *is* stabilized-interval-free *(SIF) if it does not stabilize any proper subinterval of* $[n]$. *For example* $(3, 6, 5, 4)(1, 7, 2)$ *in cycle notation, fails to be SIF because it stabilizes the interval* $[3, 6] = \{3, 4, 5, 6\}$.

Theorem 13. $\mathbf{WT}(\pi)$ *has distinct, nonzero columns if and only if* π *is SIF.*

Theorem 14. *The number of permutations* π *of* n *such that* $dim(\Delta_\pi) = k$ *is*

$$[x^{n-1-k}]2\prod_{i=2}^{n-1}(x+i).$$

That is, the generating function for permutations listed by their affine co-dimension is $2(x+2)(x+3)\ldots(x+n-1)$.

6 Types B and C

Since the Waldspurger and Meinrenken decompositions are defined for other types, it is natural to ask which phenomena we have observed will hold more generally. It seems that the connection to Abelian ideals only holds in type A. The height statistic equaling the rank in the Dedekind-MacNeille completion seems more promising.

Definition 10. *For general crystallographic root systems,* Φ, *define the Waldspurger Transform of a Weyl group element* g *to be the matrix*

$$\mathbf{WT}_\Phi(g) := (Id - R_g)C_\Phi^{-1}$$

where R_g *is the matrix of* g *in the coordinates of the simple roots of* Φ, *and* C_Φ *is the Cartan Matrix.*

If no root system is specified, we will assume type A, so that $\mathbf{WT} = \mathbf{WT}_A$ *is the Waldspurger Transform already discussed.*

Desiring to mimic type A and keep things as combinatorial as possible, consider the representation of the Weyl group $B_n = C_n$, as *centrally symmetric* permutations in $\mathfrak{S}_{2n}$ and consider the "folding map":

$$\mathcal{F} : \mathbf{WT}_{A_{2n-1}}(\mathfrak{C}\mathfrak{S}_{2\mathfrak{n}}) \longrightarrow \mathrm{Mat}_n$$

where

$$\mathcal{F}(M)_{i,j} = \begin{cases} M_{i,j} + M_{2n-i+1,j} \text{ for all } 1 \leq i, j < n \\ M_{i,j} \text{ for all } i = n, j \leq n. \end{cases}$$

Theorem 15. *$\mathcal{F}$ is a bijection between centrally symmetric Waldspurger Matrices of type A_{2n-1}, and Waldspurger Matrices of type C_n and the following diagram commutes:*

$$\begin{tikzcd} B_n \arrow[r, "\mathbf{WT}_{C_n}"] & \mathbf{WT}_{C_n}(B_n) \\ \mathfrak{C}\mathfrak{S}_{2n} \arrow[u, "\star"'] \arrow[r, "\mathbf{WT}"] & \mathbf{WT}(\mathfrak{C}\mathfrak{S}_{2\mathfrak{n}}) \subset UM_{n-1} \arrow[u, "\mathcal{F}"] \end{tikzcd}$$

Theorem 16. $\mathbf{WT}_{C_n}(\pi^\intercal) = (\mathbf{WT}_{B_n}(\pi))^\intercal$.

Acknowledgements

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References

[1] F. Ardila, F. Rincón, and L. Williams. "Positroids and non-crossing partitions". *Trans. Amer. Math. Soc.* **368** (2016), pp. 337–363. DOI.

[2] P. V. Bibikov and V. S. Zhgoon. "On tilings defined by discrete reflection groups". 2009. arXiv:0911.3919.

[3] G. Birkhoff. *Lattice Theory*. Colloquium Publications. Amer. Math. Soc., 1964, pp. 126–128.

[4] D. M. Bressoud and J. Propp. "How the alternating sign matrix conjecture was solved". *Notices Amer. Math. Soc.* **14** (1999), pp. 637–646. URL.

[5] E. Meinrenken. "Tilings defined by affine Weyl groups". *Pacific J. Math.* **242** (2009), pp. 333–343. DOI.

[6] D. I. Panyushev. "Abelian ideals of a Borel subalgebra and subsets of the Dynkin diagram". *J. Algebra* **344** (2011), pp. 197–204. DOI.

[7] J.-L. Waldspurger. "Une remarque sur les systèmes de racines". *J. Lie Theory* **17** (2007), pp. 597–603. URL.